

Physics 110B
HW#7 Solutions

#1.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E'/c & -E^2/c & -E^3/c \\ E'/c & 0 & B^3 & B^2 \\ E^2/c & B^3 & 0 & -B' \\ E^3/c & -B^2 & B' & 0 \end{pmatrix}$$

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu}$$

$$F_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & 0 & \\ & 0 & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E'/c & -E^2/c & -E^3/c \\ E'/c & 0 & B^3 & B^2 \\ E^2/c & B^3 & 0 & -B' \\ E^3/c & -B^2 & B' & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & 0 & \\ & 0 & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & 0 & \\ & 0 & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} 0 & E'/c & E^2/c & E^3/c \\ E'/c & 0 & -B^3 & -B^2 \\ E^2/c & -B^3 & 0 & B' \\ E^3/c & B^2 & -B' & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E'/c & E^2/c & E^3/c \\ -E'/c & 0 & B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B' \\ -E^3/c & -B^2 & B' & 0 \end{pmatrix}$$

#2. $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\mathcal{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$\mathcal{F}^{\nu\mu} = \frac{1}{2} \epsilon^{\nu\mu\rho\sigma} F_{\rho\sigma} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = -\mathcal{F}^{\mu\nu}$$

Thus, we see that $\mathcal{F}^{\mu\nu}$ is an antisymmetric tensor with:

$$\mathcal{F}^{00} = \mathcal{F}^{11} = \mathcal{F}^{22} = \mathcal{F}^{33} = 0$$

$$\begin{aligned}\mathcal{F}^{01} &= \frac{1}{2} \epsilon^{0100} F_{00} = \frac{1}{2} \epsilon^{0123} F_{23} + \frac{1}{2} \epsilon^{0132} F_{32} \\ &= \frac{1}{2} F_{23} - \frac{1}{2} F_{32} = -B^1\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{02} &= \frac{1}{2} \epsilon^{0213} F_{13} + \frac{1}{2} \epsilon^{0231} F_{31} = -\frac{1}{2} F_{13} + \frac{1}{2} F_{31} \\ &= -\frac{1}{2} B_1 + \frac{1}{2} (-B_2) = -B_2\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{03} &= \frac{1}{2} \epsilon^{0312} F_{12} + \frac{1}{2} \epsilon^{0321} F_{21} = \frac{1}{2} F_{12} - \frac{1}{2} F_{21} \\ &= -B^3\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{12} &= \frac{1}{2} \epsilon^{1230} F_{30} + \frac{1}{2} \epsilon^{1203} F_{03} = -\frac{1}{2} F_{30} + \frac{1}{2} F_{03} \\ &= E^3/c\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{13} &= \frac{1}{2} \epsilon^{1320} F_{20} + \frac{1}{2} \epsilon^{1302} F_{02} = \frac{1}{2} F_{20} - \frac{1}{2} F_{02} \\ &= -E^2/c\end{aligned}$$

$$\begin{aligned}\mathcal{F}^{23} &= \frac{1}{2} \epsilon^{2310} F_{10} + \frac{1}{2} \epsilon^{2301} F_{01} = -\frac{1}{2} F_{10} + \frac{1}{2} F_{01} \\ &= E/c\end{aligned}$$

With the above results and the antisymmetric condition, $\mathcal{F}^{\mu\nu} = \mathcal{F}^{\nu\mu}$, we get:

$$\boxed{\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3/c & -E^2/c \\ B^2 & -E^3/c & 0 & E/c \\ B^3 & E^2/c & -E/c & 0 \end{pmatrix}}$$

Thus, we see that $\mathcal{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ by changing: $E^i \rightarrow cB^i$ and $cB^i \rightarrow -E^i$.

$$\begin{aligned}
 \#3. \quad F^{\mu\nu}F_{\mu\nu} &= \sum_{\mu} \sum_{\nu} F^{\mu\nu}F_{\mu\nu} = F^{00}F_{00} + F^{10}F_{10} + F^{20}F_{20} + \dots \\
 &= 0 + \frac{1}{2}(-E'B' - E^2B^2 - E^3B^3 - E'B' + 0 - E^3B^3 \\
 &\quad - E^2B^2 - E^3B^3 + 0 - E'B' - E^3B^3 - E^2B^2 - E^2B^2) + 0 \\
 &= -\frac{4}{C}(E_x B_x + E_y B_y + E_z B_z) \\
 \boxed{F^{\mu\nu}F_{\mu\nu} = -\frac{4}{C}\bar{E} \cdot \bar{B}} \quad \boxed{\text{the constant of proportionality} = -\frac{4}{C}}
 \end{aligned}$$

$$\#4. \text{ Prove: } \partial_{\mu} F^{\mu\nu} = 0$$

$$\begin{aligned}
 \partial_{\mu} F^{\mu\nu} &= \partial_{\mu} (\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}) \\
 &= \partial_{\mu} (\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\partial_{\rho} A_{\sigma} - \partial_{\sigma} A_{\rho})) \\
 &= \frac{1}{2} \partial_{\mu} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma} - \frac{1}{2} \partial_{\mu} \epsilon^{\mu\nu\rho\sigma} \partial_{\sigma} A_{\rho}
 \end{aligned}$$

relabeling dummy indices

$$= \frac{1}{2} \partial_{\mu} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma} - \frac{1}{2} \partial_{\mu} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma}$$

permuting indices in epsilon

$$\begin{aligned}
 &= \frac{1}{2} \partial_{\mu} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma} + \frac{1}{2} \partial_{\mu} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma} \\
 &= \partial_{\mu} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma}
 \end{aligned}$$

$$\textcircled{1} = \epsilon^{\mu\nu\rho\sigma} \partial_{\mu} \partial_{\rho} A_{\sigma} \quad \text{Now, } \partial_{\mu} \partial_{\rho} = \partial_{\rho} \partial_{\mu}$$

relabeling dummy indices

$$= \epsilon^{\rho\nu\mu\sigma} \partial_{\rho} \partial_{\mu} A_{\sigma}$$

permuting indices in epsilon

$$\textcircled{2} = -\epsilon^{\mu\nu\rho\sigma} \partial_{\rho} \partial_{\mu} A_{\sigma} = -\epsilon^{\mu\nu\rho\sigma} \partial_{\mu} \partial_{\rho} A_{\sigma}$$

However, from lines ① and ②
we have the impossible statement:

$$\epsilon^{\mu\nu\rho\sigma} \partial_{\mu} \partial_{\rho} A_{\sigma} = -\epsilon^{\mu\nu\rho\sigma} \partial_{\mu} \partial_{\rho} A_{\sigma}$$

Thus, it follows that,

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu \partial_\rho A_\sigma = 0$$

And, thus:

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

#5. In problem 3, we found the Lorentz invariant quantity:
 (a)

$$① \quad F^{\mu\nu} F_{\mu\nu} = -\frac{4}{c} \vec{E} \cdot \vec{B}$$

In a previous problem set we also found the Lorentz invariant quantity:

$$② \quad E^2 - c^2 B^2 = \text{constant}$$

Thus, is it possible to have a frame of reference where:

$$\bar{E} = \text{constant} = A_1, \quad \bar{B} = 0 \quad \text{while in another frame} \quad \bar{E}' = 0, \quad \bar{B} = \text{constant} = A_2$$

From eq ② we have,

$$\begin{aligned} E^2 - c^2 B^2 &= E'^2 - c^2 B^2 \\ \Rightarrow A_1^2 &= -c^2 A_2^2 \leftarrow \text{not true} \end{aligned}$$

The above statement is not true thus,

No, it is not possible to have a purely electric field in one frame, while in another frame a purely magnetic field

(b) To have the following situation:

$$\bar{E} = A_1, \quad \bar{B} = A_2; \quad \bar{E}' = 0, \quad \bar{B}' = A_3$$

We will need to satisfy the invariant equations ① and ②:

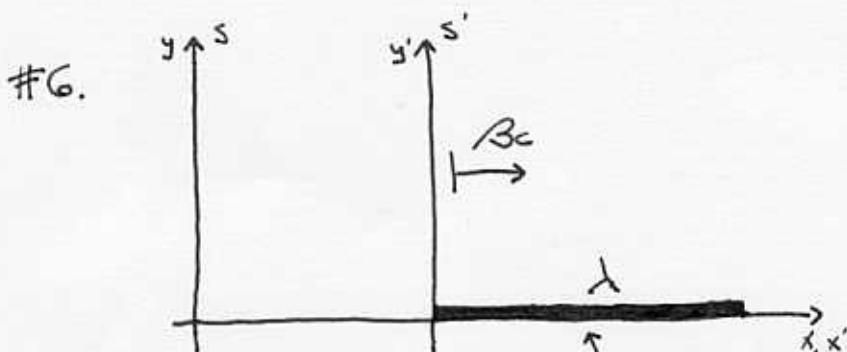
from eq①: $E^2 - c^2 B^2 = E'^2 - c^2 B'^2 = -c^2 B'^2$

Thus, $E^2 - c^2 B^2 < 0$

$$\Rightarrow \boxed{E < cB} \quad \text{is one condition that must be satisfied}$$

From eq①: $\bar{E} \cdot \bar{B} = \bar{E}' \cdot \bar{B}' = 0$

Thus, $\Rightarrow \boxed{\bar{E} \perp \bar{B}}$ and another is that the fields have to be perpendicular.



In the rest frame we use Gauss' Law to calculate the electric field at $(0, y', 0)$

$$\oint \bar{E}' \cdot d\bar{A}' = \frac{Q_{\text{enclosed}}}{\epsilon_0} \Rightarrow E' 2\pi r' \ell' = \frac{\lambda \ell'}{\epsilon_0}$$

$$\Rightarrow E' = \frac{\lambda}{2\pi\epsilon_0 r'} \hat{r}' \Rightarrow \bar{E}' = \frac{\lambda}{2\pi\epsilon_0 y'} \hat{y}, \bar{B}' = 0$$

Now, we transform the Electric and magnetic field into the lab frame:

$$E_x = E'_x = 0$$

Griffiths
eq(12.108)

$$E_y = \gamma(E'_y + vB_z) = \gamma \frac{\lambda}{2\pi\epsilon_0 y} \hat{y}$$

$$E_z = \gamma(E'_z - vB_y) = 0$$

$$B_x = B'_x = 0$$

$$B_y = \gamma(B'_y - v/c^2 E'_z) = 0$$

$$B_z = \gamma(B'_z + v/c^2 E'_y) = 0$$

Thus,

$$F^{uv} = \begin{pmatrix} 0 & 0 & -\frac{\gamma\lambda}{2\pi\epsilon_0 c y} & 0 \\ 0 & 0 & -\frac{\gamma\beta\lambda}{2\pi\epsilon_0 c y} & 0 \\ \frac{\gamma\lambda}{2\pi\epsilon_0 c y} & \frac{\gamma\beta\lambda}{2\pi\epsilon_0 c y} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

#7.

$$\frac{\bullet}{m_e} \xrightarrow{\bar{E}}$$

$$\bar{F} = e\bar{E} \Rightarrow \frac{dp}{dt} = eE$$

$$\Rightarrow \frac{d}{dt} m \gamma \beta c = eE$$

$$\frac{d}{dt} m \frac{\beta c}{\sqrt{1-\beta^2}} = eE \quad \beta = \tanh \eta$$

$$\frac{d}{dt} mc \frac{\tanh \eta}{\sqrt{1-\tanh^2 \eta}} = eE$$

$$\frac{d}{dt} mc \frac{\tanh \eta}{\operatorname{sech} \eta} = eE$$

$$\int d(mc \sinh \eta) = \int eE dt$$

$$mc \sinh \eta = eEt + C$$

at $t=0$, $\beta=0$ thus:

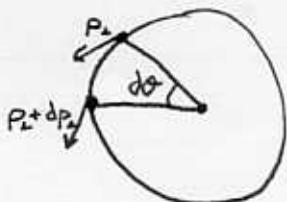
$$\tanh \gamma = 0 \Rightarrow \sinh \gamma = 0 \quad \text{So,} \quad C=0$$

Now, $mc \sinh \gamma = eEt$

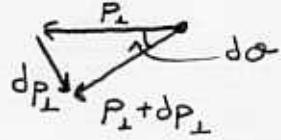
Or,

$$\boxed{\gamma = \sinh^{-1}\left(\frac{eEt}{mc}\right)}$$

8. (See Griffiths Example 12.11)



motion in the perpendicular plane \perp



dP equals approximately the arc length: $dP_{\perp} \approx p_{\perp} d\theta$

Thus,

$$F_{\text{cent}} = \frac{dP_{\perp}}{dt} = P_{\perp} \frac{d\theta}{dt} = P_{\perp} \Omega = \gamma_{\perp} m_{\perp} \beta_{\perp} c \Omega$$

where,

$$\boxed{\gamma_{\perp} \equiv \frac{1}{\sqrt{1-\beta_{\perp}^2}}}$$

• Note that we did not use mass = M since the \perp plane is not at rest with respect to the lab frame. When the particle is at rest in the \perp plane then it has mass = m_{\perp} .

$$F_{\text{cent.}} = F_{\text{Lorentz}} \Rightarrow \gamma_{\perp} m_{\perp} \beta_{\perp} c \Omega = e v_{\perp} B$$

$$\Rightarrow \gamma_{\perp} m_{\perp} \beta_{\perp} c \Omega = e c \beta_{\perp} B$$

$$\boxed{\Omega = \frac{eB}{\gamma_{\perp} m_{\perp}}}$$

Now, in the lab frame:

$$\mathbf{p} = (E, p_0, p_{\perp} \cos\theta, p_{\perp} \sin\theta)$$

We can express E and p_{\perp} as follows:

$$E = \gamma m c = \gamma_{\perp} m_{\perp} c$$

$$p_{\perp} = \gamma m \beta_{\perp} c = \gamma_{\perp} m_{\perp} \beta_{\perp} c$$

$$\text{Thus, } \mathbf{p} \cdot \mathbf{p} = E^2 - p_0^2 - p_{\perp}^2 = m^2 c^2$$

$$\gamma_{\perp}^2 m_{\perp}^2 c^2 - p_0^2 - \gamma_{\perp}^2 m_{\perp}^2 \beta_{\perp}^2 c^2 = m^2 c^2$$

$$\gamma_{\perp}^2 m_{\perp}^2 c^2 (1 - \beta_{\perp}^2) = m^2 c^2 + p_0^2$$

So,

$m_{\perp} = \sqrt{m^2 + p_0^2/c^2}$